

(see Fig. 11). Even for the extreme values of $R/r_o = 1.1$ the contribution of the omitted term is considerable.

Discussion by Odysseus Michaelides⁴ and George Gazetas⁵

The authors present a closed-form analytical solution to the plane-strain axisymmetric problem of a vertically vibrating pile slice, surrounded by soil whose shear modulus G increases with radial distance r . Such a solution would serve as a key to obtaining (in an approximate simplified way) the dynamic vertical response of a pile in nonlinear soil. While numerical solutions to this problem had been published by Dotson and Veletsos (1987) and Veletsos and Dotson (1987, 1990) who used a variety of radially increasing functions for $G(r)$, the authors introduce a parabolic function, (3), which would seem to better fit experimental data (e.g. Bouckovalas et al. 1992). Moreover, the authors obtain a closed-form solution to the governing differential equation of motion, (10)—a potentially interesting development, indeed.

Unfortunately, this differential equation, (10), which the authors derive in the paper instead of taking it from the published literature (Veletsos and Dotson 1987; Gazetas and Dobry 1984) apparently is irrelevant to the studied problem. Eq. (10) applies only to a one-dimensional medium, such as an inhomogeneous bar of uniform cross section or a soil column, and not to the two-dimensional axisymmetric geometry of a vertically oscillating pile.

The correct equation for this latter case is

$$G^*(r) \frac{\partial^2 w}{\partial r^2} + \left[\frac{dG^*(r)}{dr} + \frac{G^*(r)}{r} \right] \frac{\partial w}{\partial r} = \frac{\partial^2 w}{\partial t^2} \quad (52)$$

Comparing (52) with (10) of the paper, one notices that the term

$$\frac{G^*(r)}{r} \frac{\partial w}{\partial r} \quad (53)$$

present in (52) is missing from (10). Yet, this term makes all the difference. One can immediately see that if $G^*(r)$ were taken as constant, (47) would reduce to the Bessel-type equation

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} = \frac{\rho}{G^*} \frac{\partial^2 w}{\partial t^2} \quad (54)$$

(e.g. Novak et al. 1987; Graff 1975), whereas (10) reduces to the classical one-dimensional wave equation

$$\frac{\partial^2 w}{\partial r^2} = \frac{\rho}{G^*} \frac{\partial^2 w}{\partial t^2} \quad (55)$$

which is not applicable to the problem at hand.

It is easy to find out where the error was committed in deriving (10). Considering an elemental ring of infinitesimal width Δr , the authors obtain in (8) the expression for the vertical shear force, τ_2 , on the outer surface by multiplying the shear-stress expression by $2\pi r$. Instead, they obviously should have multiplied by $2\pi(r + \Delta r)$. In other words, τ_2 should read:

$$\tau_2 = - \left[G(r) + \frac{dG^*(r)}{dr} \Delta r \right] \left(\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial r^2} \cdot \Delta r \right) \cdot 2\pi(r + \Delta r) \quad (56)$$

Then, if (56) were combined with (7) and (9), the correct equation of motion, (52), would be derived.

Unfortunately, with the parabolic variation of modulus $G(r)$ introduced in the paper, (3), (52) cannot be solved in closed form (Abramowitz and Stegun 1972). And, of course, the hypergeometric-function solution of (22) is neither a solution to the governing differential equation, nor is it relevant to the analysis of the vertical response of a pile.

The discussor made three other remarks on the paper: First, (14) is not consistent with (3). The latter can be written as

$$f(\xi) = 1 - x^2 \quad (57)$$

where

$$x = m \left(\xi - \frac{R}{r_o} \right) \quad (58)$$

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and not

$$x = m \left(\frac{R}{r_o - \zeta} \right) \quad (59)$$

as given in the paper.

Second, in Fig. 3, the authors seem to imply that the "homogeneous" solution was, or can be, derived as a special case ($G^* = \text{constant}$) of their solution. This is not correct. Because in this case their solution reduces to (55) (as already explained) which leads to the purely imaginary impedance

$$k_w = i\omega c_w \quad (60)$$

where the "viscous dashpot" constant is given by (Gazetas 1987)

$$c_w = \rho V_s(2\pi r_o) = \sqrt{\rho G}(2\pi r_o) \quad (61)$$

In the notation of the paper, the preceding results imply

$$\alpha = 0 \quad \text{and} \quad \beta = 2 \quad (62a,b)$$

Instead, Fig. 3 shows a nonzero α and a frequency-dependent β . Obviously, then, the "homogeneous" solution plotted in Fig. 3 was not (and cannot be) derived from the results of the authors.

Third, the authors' damping factor β in Fig. 3 decreases monotonically to achieve a value of about 1.20 at $a_i = 4$. In reality, in any laterally inhomogeneous or homogeneous soil, β approaches asymptotically the value of 2.

CONCLUSION

Regrettably, the results of the paper must not be used in analyzing the dynamic response of piles.

APPENDIX. REFERENCES

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Closure by Hans Vaziri⁶

The primary issue resulting in the discussions presented by Makris, and Michaelides and Gazetas pertains to the omission of $(1/r)G(r)(\partial w/\partial r)$ term in (10). We concur with both contributors on the omission cited and its significance. We regret this oversight and appreciate the important contribution that the writers have made in identifying this problem.

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